Duration : 55 minutes

Analysis II
Midterm
IN/SC
Spring 2024

Questions

For the **multiple choice** questions, we give

$+3$ points if your answer is correct,

0 points if you give no answer or more than one,

-1 if your answer is incorrect.

For the **true/false** questions, we give

$+1$ points if your answer is correct,

0 points if you give no answer or more than one,

-1 points if your answer is incorrect.

Part I: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 : The solution $y(x)$ of the differential equation

$$x\, y'(x) = \frac{\big(y(x)\big)^2}{\ln(x) + 1}$$

on the interval $(1, +\infty)$ with initial condition $y(e) = -\dfrac{1}{2\ln(2)}$ also satisfies

- ☐ $y(e^3) = -\dfrac{1}{\ln(2)}$
- ☐ $y(e^3) = -\dfrac{1}{2\ln(2)}$
- ☐ $y(e^3) = -\dfrac{1}{3\ln(2)}$
- ☐ $y(e^3) = \dfrac{1}{4\ln(2)}$

Question 2 : The solution $y(x)$ of the differential equation

$$y''(x) - 3y'(x) + 2y(x) = 2x^2 - 2x$$

with initial conditions $y(0) = 2$ and $y'(0) = 2$ also satisfies

- ☐ $y(1) = 5 - 2e^2$
- ☐ $y(1) = \frac{5}{4} + 2e$
- ☐ $y(1) = 5 + 2e - 2e^2$
- ☐ $y(1) = 5$

Question 3 : The subset

$$A = \left\{(x, y) \in \mathbb{R}^2 : x^2 \leq 1 \text{ and } y \leq x^2\right\} \subset \mathbb{R}^2$$

- ☐ is closed and not bounded
- ☐ is closed and bounded
- ☐ is not bounded and not closed
- ☐ is bounded and not closed

Question 4 : Let $f : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}$ be the function defined by

$$f(x, y) = \frac{y^2 \sin^2(x)}{x^4 + y^2}.$$

Then

- ☐ $\lim\limits_{(x,y)\to(0,0)} f(x, y) = \dfrac{1}{2}$
- ☐ $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist
- ☐ $\lim\limits_{(x,y)\to(0,0)} f(x, y) = 0$
- ☐ $\lim\limits_{(x,y)\to(0,0)} f(x, y) = \dfrac{\pi^2}{4}$

Question 5: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Then

- ☐ f is continuous at $(0,0)$, $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$ exist, but f is not differentiable at $(0,0)$
- ☐ f is continuous at $(0,0)$, but $\dfrac{\partial f}{\partial x}(0,0)$ does not exist
- ☐ f is not continuous at $(0,0)$
- ☐ f is differentiable at $(0,0)$

Question 6: Let $\boldsymbol{g} : \mathbb{R}^3 \to \mathbb{R}^3$ be the function defined by

$$\boldsymbol{g}(x,y,z) = \big(xy,\ xz,\ yz\big)^{\mathrm{T}}$$

and let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function defined by

$$f(u,v,w) = uvw\,.$$

Then the composition $h = f \circ \boldsymbol{g} : \mathbb{R}^3 \to \mathbb{R}$ satisfies

- ☐ $\dfrac{\partial h}{\partial z}(-1,0,1) = 0$
- ☐ $\dfrac{\partial h}{\partial z}(-1,0,1) = -1$
- ☐ $\dfrac{\partial h}{\partial z}(-1,0,1) = 1$
- ☐ $\dfrac{\partial h}{\partial z}(-1,0,1) = 2$

Question 7: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = e^{x^2+4x-2y}\,.$$

The second order Taylor polynomial for f around $(0,0)$ is

- ☐ $p_2(x,y) = 1 + 4x - 2y + 9x^2 - 4xy + 2y^2$
- ☐ $p_2(x,y) = 1 + 4x - 2y + 9x^2 - 8xy + 2y^2$
- ☐ $p_2(x,y) = 1 + 4x - 2y + 18x^2 - 8xy + 4y^2$
- ☐ $p_2(x,y) = 1 + 4x - 2y + 18x^2 - 16xy + 4y^2$

Part II: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 8: Let $A = \mathbb{R}^3 \backslash \{(0,0,0)\}$. Then the boundary ∂A is the empty set.

☐ TRUE ☐ FALSE

Question 9: Let $f : \mathbb{R} \to (0,1)$ be a function such that $\lim_{t \to 0} f(t) = L > 0$, then

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{f\big(x^2 + y^2\big)} = 0\,.$$

☐ TRUE ☐ FALSE

Question 10: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function and let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by $g(x,y) = \big(f(x,y)\big)^2$. If f is not differentiable at $(x,y) = (0,0)$, then g is not differentiable at $(x,y) = (0,0)$.

☐ TRUE ☐ FALSE

Question 11: Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function of class C^3. Then

$$\frac{\partial^3 f}{\partial x^2 \partial y}(x,y,z) = \frac{\partial^3 f}{\partial x \partial y \partial x}(x,y,z)\,, \qquad \text{for all } (x,y,z) \in \mathbb{R}^3.$$

☐ TRUE ☐ FALSE